THE UNIVERSAL EQUATIONS AND PARAMETRIC APPROXIMATIONS IN THE THEORY OF THE LAMINAR BOUNDARY LAYER

(UNIVERSAL'NYE URAVNENIIA I PARAMETRICHESKIE PRIBLIZHENIIA V TEORII LAMINARNOGO POGRANICHNOGO SLOIA)

PMM Vol.29, № 1, 1965, pp.70-87

L.G. LOITSIANSKII
(Leningrad)

(Received October 3, 1964)

Existing analytic methods of solution of the equations of the laminar boundary layer are based on the representation of these solutions in the form of power series of certain parameters, either characterizing the external distribution only (Falkner [1], Görtler [2], Shkadov [3 and 4] and others), or else taking account in addition of the development of the boundary layer (Loitsianskii [5]). In all these methods the determination of the coefficients in the power series reduces to the integration of systems of ordinary differential equations possessing the property of universality in the sense that neither these equations themselves, nor the corresponding boundary conditions, depend upon the conditions of a given particular problem, and consequently they can be integrated numerically in advance once and for all, and the results tabulated. The effectiveness of the application of such methods is limited by the speed of convergence of the series, which decreases sharply in the neighborhood of the point of sepatation of the boundary layer, this being a singular point of the equations at which the power series are inapplicable and must be replaced by series of another type, containing together with power terms, logarithmic terms also.

One of the ways of improving the analytical methods can be that recommended in the present paper: the method of reducing the fundamental equations of the boundary layer to universal form, in the sense already specified. The possibility of such universalization of the boundary layer equations themselves by transferring the parameters expressing the influence of the external conditions, characteristic of each particular problem, to the number of independent variables, is demonstrated in examples of the boundary layer equations in incompressible liquid, and in uniform gases and gases under dissociative equilibrium, at large velocities of motion.

The fundamental significance of the method described consists in the replacement of the series by the use of tables composed once and for all by means of direct numerical integration of the universal partial differential equations.

The difficulties mentioned above, connected with the presence of a singularity at the point of separation, whilst emerging anew as a result of the increase in the number of independent variables in the differential equations, are at the same time transferred from the field of analysis to the field of machine computation techniques, and, because of the need to carry out this whole calculation only once in the compilation of the tables, this can scarcely be counted amongst the essential shortcomings of the method. On the other hand, it is necessary to bear in mind the fact that the increase in the number of arguments carries the inconvenience associated with the use of tables with a large number of "entries". This forces us to consider the question of the choice of such a sequence of parameters that the use of the first two or three of them should lead right away to acceptable accuracy of the results. To serve as such a system of parameters for the boundary layer in incompressible fluid, in the present paper we shall take a set of dimensionless quantities, expressing in terms of powers of the external velocity, its successive derivatives with respect to the longitudinal coordinate, and also by power series the relationship of the square of the momentum thickness in the given section of boundary layer to the kinematic coefficient of viscosity. The first term of this set comprises the "shape parameter", used in the various approximate one-parameter methods, based on the application of the integral momentum condition and discriminating by its choice between this or that form of velocity profile in the boundary layer cross-sections [6 and 7]. Moreover, even the first approximation, calculated by means of integrating the universal equation with one parameter, leads to the same result as the one-parameter method, if in the latter we take for the "competing" velocity profile in the boundary layer sections the class of exact solutions of the boundary layer equations corresponding to a linear distribution of velocity outside the boundary layer [8 and 9]. As is well known, this approximation which will be called the "one-parameter" solution, depicts in a completely satisfactory manner the actual motion in the boundary layer for the majority of velocity distributions encountered at the external surface of the boundary layer. It may therefore be thought that application of the "two-parameter", or in the extreme case, the "three-parameter" approximation to the solution of the universal equations will prove to be completely satisfactory for a broad family of problems, and therefore it will not be necessary to make use of the tables with an excessively large number of entries. This fact is confirmed by the comparison with the exact solution, made in this paper, for the problem concerning the boundary layer in incompressible fluid with a sinusoidal distribution of velocities on the outer surface [10].

The set of form-parameters here proposed, thanks to their inclusion of the determination of the momentum thickness enables one to obtain, although only approximately, the solution of the well-known boundary layer theory problem of "continuation", i.e. calculation of the influence ot the previous history of the flow in the boundary layer on its development in the portions of the layer further downstream.

As is shown in the final sections of this paper, universalization of the equations of the laminar boundary layer is accomplished not only in the case of isothermal flow of an incompressible fluid, but also in the more general cases of the boundary layer in a stream of homogeneous gas, and also of gas in dissociative equilibrium, with high velocities of motion.

The enumerated cases are not the only cases permitting universalization of the equations and the obtaining of their one and multi-parameter solutions. In particular, universal equations of this type can be derived from the equations of nonequilibrium boundary layers, the boundary layers in magnetohydrodynamics and other physically more complex problems in the theory of the laminar boundary layer. What is more, the same method of universalization of the equations and boundary conditions can be useful in the solution of problems, relating not only to the theory of the boundary layer, but also to the other problems where the integration of nonlinear partial differential equations of parabolic type is necessary (for example, to nonlinear problems of heat conduction).

1. The isothermal boundary layer in incompressible fluid; the differential equation for the reduced stream function. The laminar boundary layer equation in incompressible fluid and the conditions for isothermal flow reduce to an equation for the stream function $\psi(x,y)$

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=U\frac{dU}{dx}+\nu\frac{\partial^3\psi}{\partial y^3} \tag{1.1}$$

$$\psi=\frac{\partial\psi}{\partial y}=0\quad\text{for }y=0,\quad \frac{\partial\psi}{\partial y}=u\to U(x)\quad\text{for }y\to\infty,\quad \frac{\partial\psi}{\partial y}=u_0(y)\quad\text{for }x=x_0$$

Here we adopt the usual boundary layer theory notation : x, y are the longitudinal and perpendicular coordinates, u is the longitudinal velocity, $u_0(y)$ the distribution of velocity at a certain given section of the boundary layer $(x = x_0)$; $U(x)$ is the longitudinal velocity at its outer face, and ν is the coefficient of kinematic viscosity.

Let us transform from the variables x, y, $\psi(x, y)$ to the new variables x, ξ, $\Phi(x, \xi)$, by substituting in (1.1) (1.2)

$$x = x, \quad y = \left(\frac{\delta^{**}}{B_0}\right)\xi, \quad \psi = \left(\frac{U\delta^{**}}{B_0}\right)\Phi(x, \xi) \qquad \left(\delta^{**}(x) = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy\right)$$

Here δ^{**} is the momentum thickness; B_0 is a normalizing constant which is determined later.

Let us carry out the transformation (1.2) in Equation (1.1), making use also of the well-known momentum equation, written in the following of its various forms (primes here and in what follows denote differentiation with respect to x) :

$$\frac{\delta^{**\prime}}{\delta^{**}} = \frac{U'F}{2Uf}, \qquad z^{**\prime} = \frac{F}{U}, \qquad f' = \frac{U'}{U}F + \frac{U''}{U'}f \tag{1.3}$$

Here we have introduced the conventional boundary layer theory notation

$$F = 2[\zeta - (2 + H)f], \qquad \zeta = \left[\frac{\partial(u/U)}{\partial(y/\delta^{**})}\right]_{y=0}, \qquad H = \frac{\delta^*}{\delta^{**}}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy, \qquad f = \frac{U'\delta^{**2}}{\nu} = U'z^{**}, \qquad z^{**} = \frac{\delta^{**}}{\nu} \tag{1.4}$$

Let us reduce Equation (1.1) to the form (the dot denotes differentiation with respect to ξ) (1.5)

$$\frac{\partial^3\Phi}{\partial\xi^3} + \frac{F + 2f}{2B_0^2}\Phi\frac{\partial^2\Phi}{\partial\xi^2} + \frac{f}{B_0^2}\left[1 - \left(\frac{\partial\Phi}{\partial\xi}\right)^2\right] = \frac{U}{B_0^2U'}f\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial x\,\partial\xi} - \frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial\xi^2}\right)$$

$$\Phi = \Phi^{\cdot} = 0 \ \text{ for } \xi = 0, \quad \Phi^{\cdot} \to 1 \ \text{ for } \xi \to \infty, \quad \Phi^{\cdot} = \Phi_0^{\cdot}(\xi) \ \text{ for } x = x_0$$

Making use of the "reduced" stream function $\Phi(x, \xi)$ we shall have, in accordance with (1.4),

$$\frac{u}{U} = \frac{\partial\Phi}{\partial\xi}, \qquad \zeta = B_0\left(\frac{\partial^2\Phi}{\partial\xi^2}\right)_{\xi=0}, \qquad H = \frac{1}{B_0}\int_0^\infty \left(1 - \frac{\partial\Phi}{\partial\xi}\right) d\xi \tag{1.6}$$

For definiteness let us take for the function $\Phi_0(\xi)$ the simplest self-similar solution of the boundary layer equation, corresponding to conditions

$$U = \text{const}, \qquad U' = 0, \qquad f = 0 \tag{1.7}$$

(quantities relating to these conditions will hereafter be denoted by the index zero) and choose the normalizing constant B_0 so that Equation (1.5) in conditions (1.7) coincides with Blasius' equation (*)

*) For the function $\Phi_0(\xi)$ we could take any self-similar solution of Equation (1.5), whilst the normalizing constant B_0 is chosen so that this equation coincides with the well-known Falkner-Skan equation in Hartree's form; in this connection see also [5].

$$\Phi''' + \Phi_0\Phi_0'' = 0, \qquad \Phi_0 = \Phi_0' = 0 \quad \text{for } \xi = 0, \qquad \Phi_0' \to 1 \quad \text{for } \xi \to \infty \tag{1.8}$$

For this we must set

$$F_0 = 2\zeta_0 = 2B_0^2 \tag{1.9}$$

Hence from the well-known solution of Blasius' problem it follows that

$$B_0 = \Phi_0''(0) = 0.470 \tag{1.10}$$

Making use of the definition of the momentum thickness (1.2) and making the substitution of variables in it, we obtain the general relation

$$\int_0^\infty \Phi'(1-\Phi')\,d\xi = \int_0^\infty \Phi_0'(1-\Phi_0')\,d\xi = B_0 \tag{1.11}$$

which enables us to obtain a closed expression for the quantity H in terms of the limiting value of the difference of the functions $\Phi(x,\xi)$ and $\Phi_0(\xi)$ as $\xi \to \infty$.

We have

$$H = \frac{\delta^*}{\delta^{**}} = \int_0^\infty (1-\Phi')\,dy \Big/ \left(\int_0^\infty \Phi'(1-\Phi')\,dy\right) = \frac{1}{B_0}\int_0^\infty (1-\Phi')\,d\xi =$$

$$= \frac{1}{B_0}\int_0^\infty (1-\Phi_0')\,d\xi + \frac{1}{B_0}\int_0^\infty (\Phi_0' - \Phi')\,d\xi$$

Assuming the separate existence of the last two integrals, we obtain the required expression H

$$H = H_0 + (1/B_0)(\Phi_0 - \Phi)_{\xi\to\infty}, \qquad H_0 = 2.592 \tag{1.12}$$

2. Universal equation of the boundary layer in an incompressible fluid. The solution of Equation (1.5), containing the arbitrary function $U(x)$ and its first derivative, is a function of the variables x, ξ and a functional of U

$$\Phi = \Phi(x, \xi; \{U\}) \tag{2.1}$$

Let us introduce for consideration the infinite combination of form-parameters [5]

$$f_k = U^{k-1}(d^kU/dx^k)\,z^{**k}, \qquad z^{**} = \delta^{**2}/\nu \qquad (k = 1, 2, \ldots) \tag{2.2}$$

the first of which f_1 coincides with the usual form-parameter f of the one-parameter methods. By virtue of arbitrariness of the function $U(x)$ the form-parameters $f_k(x)$ form a system of independent functions satisfying the recurrent ordinary differential equation

$$(U/U')\,f_1 f_k' = [(k-1)\,f_1 + kF]\,f_k + f_{k+1} \tag{2.3}$$

which is easily derived from (2.2) by means of direct differentiation and use of (1.3).

We shall show that the equation and the boundary conditions (1.5) can be satisfied by Expression

$$\Phi = \Phi(\xi; \{U\}) = \Phi(\xi; f_1, f_2, \ldots) \tag{2.4}$$

which does not contain x explicitly. In other words, we shall show that, making use of expressions, in accordance with (2.2), for the "reduced" stream function Φ and the normal coordinate ξ, the influence of the external flow on the flow in the given section of the boundary layer can be expressed by means of the combination of form-parameters (2.2) only. For we note that, according to (2.4), Equations (1.6) and (1.12) may be rewritten thus:

$$\zeta = \zeta(f_1, f_2, \ldots), \qquad H = H(f_1, f_2, \ldots), \qquad F = F(f_1, f_2, \ldots) \quad (2.5)$$

so that the right-hand side of the recurrence relation (2.3) is also a function of the pqrameters f_1, f_2, ... We denote this as follows:

$$[(k-1) f_1 + kF] f_k + f_{k+1} = \theta_k (f_1, f_2, \ldots) \quad (2.6)$$

Let us choose the form-parameters f_k as the new independent variables and let us carry out in Equation (1.5) the substitution of the differential operator according to Formula

$$\frac{\partial}{\partial x} = \sum_{k=1}^{\infty} f_k' \frac{\partial}{\partial f_k} = \frac{U'}{U f_1} \sum_{k=1}^{\infty} \theta_k \frac{\partial}{\partial f_k} \quad (2.7)$$

following directly from (2.3). We shall have

$$\frac{\partial^3 \Phi}{\partial \xi^3} + \frac{F + 2f_1}{2B_0^2} \Phi \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{f_1}{B_0^2}\left[1 - \left(\frac{\partial \Phi}{\partial \xi}\right)^2\right] = \frac{1}{B_0^2} \sum_{k=1}^{\infty} \theta_k \left(\frac{\partial \Phi}{\partial \xi} \frac{\partial^2 \Phi}{\partial \xi \, \partial f_k} - \frac{\partial \Phi}{\partial f_k} \frac{\partial^2 \Phi}{\partial \xi^2}\right) \quad (2.8)$$

$$\Phi = \Phi' = 0 \quad \text{for } \xi = 0, \qquad \Phi \to 1 \quad \text{for } \xi \to \infty$$

$$\Phi = \Phi_0(\xi) \quad \text{for } f_1 = f_2 = \ldots = 0$$

This equation in the case of the isothermal boundary layer in incompressible fluid serves in fact for the fundamental universal equation, which has been referred to in this paper.

In fact, in this nonlinear third order partial differential equation, as also in the boundary conditions, there are no magnitudes characterizing the actual specified problem. Equations (2.8) are one and the same for all problems in the theory of the isothermal boundary layer in incompressible fluid, in which the velocity distribution on the outer boundary is continuous and admits the existence of successive derivatives at all points of its range. They all reduce, accordingly, to numerical integration of Equation (2.8) once and for all and the construction of tables for the dependence of the reduced stream function Φ and its derivative $\Phi' = u/U$ on ξ and the form-parameters f_1, f_2,.... For this purpose, eventually, we have to use an electronic computer. The function F appearing in the equation is expressed in terms of the functions ζ and H, which can be determined only after integration of Equation (2.8). This circumstance does not pose any fundamental difficulties in constructing the program for integrating Equation (2.8) on an electronic computer.

Since after this the function $\Phi(\xi; f_1, f_2, \ldots)$ is determined once and for all, the solution of any actual problem reduces to investigation of the dependence of $\delta^{**}(x)$ or $z^{**}(x)$, characterizing the particular problem. For this we need to integrate the ordinary nonlinear first order differential equation

$$\frac{dz^{**}}{dx} = \frac{F(f_1, f_2, \ldots)}{U(x)} = \frac{F(U' z^{**}, U U'' z^{**2}, \ldots)}{U(x)} \quad (2.9)$$

Further it will be demonstrated that this is the only step of computation, requiring to be carried out for each given boundary layer calculation; it can be reduced to a simple quadrature to a sufficient degree of accuracy.

3. The one-parameter solution of the universal equation. In the solution of the universal equation (2.8) there naturally occur a modest number of form-parameters to be fitted.

For the "one-parameter" solution we shall, in what follows, have in mind the solution corresponding to zero values of all the form-parameters except the first. For the "two-parameter" solution, zero values attach to all the form-parameters except the first two, and so on. Let us agree to denote these solutions by the corresponding number, placed superscript in brackets. Thus, setting

$$f_2 = f_3 = \ldots = 0 \tag{3.1}$$

we obtain the basic universal equation in the "one-parameter" approximation

$$\frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} + \frac{F^{(1)} + 2f_1}{2B_0^2} \Phi^{(1)} \frac{\partial^2 \Phi^{(1)}}{\partial \xi^2} + \frac{f_1}{B_0^2}\left[1 - \left(\frac{\partial \Phi^{(1)}}{\partial \xi}\right)^2\right] = \frac{1}{B_0^2} F^{(1)} f_1 \left(\frac{\partial \Phi^{(1)}}{\partial \xi}\frac{\partial^2 \Phi^{(1)}}{\partial \xi\, \partial f_1} - \frac{\partial \Phi^{(1)}}{\partial f_1}\frac{\partial^2 \Phi^{(1)}}{\partial \xi^2}\right) \tag{3.2}$$

$$\Phi^{(1)} = \Phi^{\cdot(1)} = 0 \quad \text{for } \xi = 0, \quad \Phi^{\cdot(1)} \to 1 \quad \text{for } \xi \to \infty, \quad \Phi^{(1)} = \Phi_0(\xi) \quad \text{for } f_1 = 0$$

Condition (3.1) in the exact formulation corresponds to the boundary layer with a linear distribution of velocity on the outer boundary. The function $\Phi^{(1)}(\xi; f_1)$ and its derivatives, and consequently also the auxiliary functions $\zeta^{(1)}(f_1)$, $H^{(1)}(f_1)$ and $F^{(1)}(f_1)$ could be obtained by making use of the one-parameter class of exact solutions due to Howarth for "one-slope" [8] distributions of external velocity, if we eliminate the parameter appearing in these solutions.

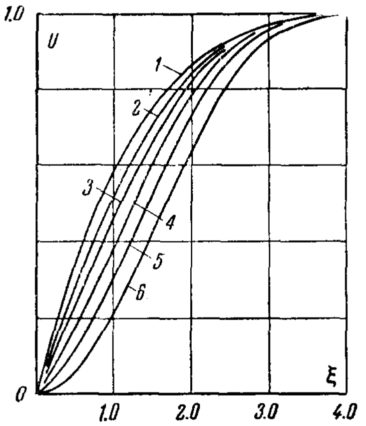

Fig. 1

A computational experiment of this type shows that preference must be given to direct numerical integration of the universal equation (3.2). Such an integration in the case of one form-parameter was carried out at the Leningrad Computation Center of the Academy of Sciences, USSR, on the computer БЭСМ-2 (BESM-2). We shall show some of the results.

In Table 1 are placed the values of the dimensionless velocity $u / U = \Phi^{\cdot(1)}(\xi, f_1)$ for a series of values of ξ and f_1. Fig.1 shows graphs of this dimensionless velocity; curves 1,... 6 correspond to values of the form-parameter f_1 = 0.0854, 0.04, 0, –0.04, –0.07, –0,0845 . In Table 2 are given values of $\zeta^{(1)}$ and $H^{(1)}$, and also $F^{(1)}$, as a function of f_1. The first two of these quantities serve for computation of the coefficient of local viscosity c_f and the displacement thickness δ^*, and the latter for determination of the momentum thickness δ^{**} and of the distribution $f_1(x)$ which plays a subsidiary role. The corresponding curves are shown in Figs. 2 and 3.

Using for the one-parameter method the first of the successive approximate solutions of the universal equation (2.8) is better founded from the theoretical point of view than the old, purely intuitive one-parameter methods. As is shown by comparative calculations, in particular, the example

ξ \ f_1	—0.0845	—0.0835	—0.08	—0.07	—0.06	—0.05	—0.04	—0.03	—0.02	—0.01
0	0	0	0	0	0	0	0	0	0	0
0.1	0.0047	0.0064	0.0103	0.0175	0.0230	0.0277	0.0320	0.0361	0.0399	0.0435
0.2	0.0132	0.0166	0.0242	0.0381	0.0486	0.0577	0.0659	0.0735	0.0807	0.0875
0.3	0.0255	0.0305	0.0416	0.0618	0.0769	0.0898	0.1015	0.1122	0.1223	0.1319
0.4	0.0415	0.0480	0.0626	0.0886	0.1078	0.1241	0.1387	0.1521	0.1647	0.1766
0.5	0.0612	0.0692	0.0869	0.1182	0.1411	0.1604	0.1775	0.1931	0.2077	0.2216
0.6	0.0844	0.0938	0.1145	0.1506	0.1767	0.1984	0.2176	0.2351	0.2514	0.2667
0.7	0.1110	0.1218	0.1452	0.1856	0.2144	0.2381	0.2590	0.2779	0.2953	0.3117
0.8	0.1408	0.1528	0.1787	0.2228	0.2539	0.2793	0.3014	0.3213	0.3396	0.3566
0.9	0.1737	0.1868	0.2149	0.2621	0.2949	0.3215	0.3445	0.3650	0.3838	0.4012
1.0	0.2093	0.2234	0.2534	0.3032	0.3373	0.3646	0.3881	0.4088	0.4277	0.4452
1.2	0.2876	0.3033	0.3362	0.3892	0.4245	0.4521	0.4754	0.4958	0.5141	0.5308
1.4	0.3731	0.3898	0.4241	0.4778	0.5125	0.5390	0.5610	0.5799	0.5966	0.6117
1.6	0.4625	0.4794	0.5237	0.5658	0.5982	0.6225	0.6422	0.6588	0.6733	0.6862
1.8	0.5522	0.5687	0.6014	0.6496	0.6787	0.6999	0.7167	0.7306	0.7425	0.7529
2.0	0.6386	0.6539	0.6838	0.7265	0.7513	0.7689	0.7826	0.7936	0.8029	0.8108
2.2	0.7181	0.7317	0.7579	0.7939	0.8141	0.8281	0.8386	0.8469	0.8537	0.8594
2.4	0.7881	0.7997	0.8215	0.8505	0.8662	0.8767	0.8843	0.8903	0.8949	0.8988
2.6	0.8469	0.8562	0.8735	0.8958	0.9073	0.9148	0.9201	0.9241	0.9271	0.9295
2.8	0.8939	0.9011	0.9141	0.9303	0.9384	0.9435	0.9469	0.9494	0.9512	0.9525
3.0	0.9296	0.9348	0.9442	0.9554	0.9608	0.9640	0.9661	0.9676	0.9685	0.9692
3.2	0.9553	0.9590	0.9653	0.9727	0.9761	0.9780	0.9792	0.9800	0.9804	0.9807
3.4	0.9729	0.9753	0.9794	0.9840	0.9860	0.9871	0.9878	0.9881	0.9883	0.9883
3.6	0.9843	0.9858	0.9883	0.9911	0.9922	0.9928	0.9931	0.9933	0.9933	0.9932
3.8	0.9914	0.9922	0.9937	0.9952	0.9958	0.9961	0.9963	0.9963	0.9963	0.9962
4.0	0.9955	0.9960	0.9968	0.9976	0.9979	0.9980	0.9981	0.9981	0.9981	0.9980
4.2	0.9977	0.9980	0.9984	0.9988	0.9990	0.9990	0.9990	0.9990	0.9990	0.9989
4.4	0.9989	0.9991	0.9992	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995
4.6	0.9995	0.9996	0.9997	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9997
4.8	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
5.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9999
5.2	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
5.4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
5.6	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
5.8	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
6.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

worked out in Section 5, the results obtained on the basis of Table 2 are a good recommendation for the method described in the present Section.

In the accepted one-parameter approximation Equation (2.9) will have the form

$$\frac{dz^{**}}{dx} = \frac{F^{(1)}(f_1)}{U(x)} = \frac{F^{(1)}(U'z^{**})}{U(x)} \tag{3.3}$$

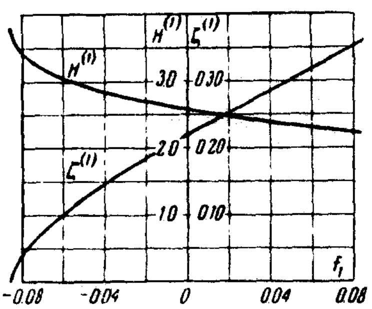

Fig. 2

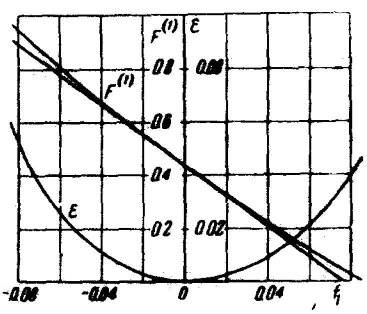

Fig. 3

Table 1

0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.0854
0	0	0	0	0	0	0	0	0	0
0.0470	0.0504	0.0537	0.0570	0.0602	0.0634	0.0666	0.0698	0.0730	0.0748
0.0939	0.1003	0.1066	0.1126	0.1186	0.1246	0.1305	0.1364	0.1424	0.1457
0.1408	0.1498	0.1584	0.1669	0.1752	0.1834	0.1916	0.1998	0.2081	0.2128
0.1876	0.1987	0.2094	0.2198	0.2300	0.2401	0.2501	0.2601	0.2704	0.2761
0.2343	0.2470	0.2593	0.2712	0.2829	0.2944	0.3059	0.3174	0.3291	0.3358
0.2806	0.2947	0.3081	0.3211	0.3339	0.3465	0.3591	0.3716	0.3844	0.3917
0.3266	0.3416	0.3558	0.3696	0.3831	0.3964	0.4096	0.4228	0.4364	0.4442
0.3720	0.3876	0.4022	0.4164	0.4303	0.4440	0.4575	0.4713	0.4851	0.4932
0.4168	0.4325	0.4473	0.4617	0.4756	0.4893	0.5029	0.5166	0.5307	0.5388
0.4607	0.4764	0.4910	0.5052	0.5189	0.5324	0.5458	0.5593	0.5732	0.5813
0.5453	0.5601	0.5738	0.5868	0.5995	0.6119	0.6242	0.6366	0.6495	0.6572
0.6245	0.6376	0.6495	0.6609	0.6718	0.6825	0.6931	0.7038	0.7150	0.7218
0.6968	0.7078	0.7176	0.7269	0.7357	0.7443	0.7528	0.7614	0.7706	0.7763
0.7611	0.7699	0.7774	0.7845	0.7912	0.7976	0.8039	0.8104	0.8174	0.8218
0.8168	0.8233	0.8287	0.8337	0.8383	0.8427	0.8470	0.8514	0.8562	0.8595
0.8634	0.8679	0.8714	0.8746	0.8775	0.8801	0.8827	0.8853	0.8882	0.8903
0.9011	0.9040	0.9061	0.9078	0.9093	0.9106	0.9117	0.9128	0.9141	0.9152
0.9307	0.9323	0.9333	0.9340	0.9344	0.9347	0.9348	0.9348	0.9348	0.9351
0.9530	0.9537	0.9540	0.9540	0.9538	0.9534	0.9528	0.9521	0.9512	0.9509
0.9691	0.9694	0.9692	0.9688	0.9683	0.9675	0.9666	0.9654	0.9640	0.9632
0.9804	0.9804	0.9800	0.9795	0.9788	0.9779	0.9768	0.9755	0.9738	0.9727
0.9880	0.9878	0.9874	0.9869	0.9862	0.9854	0.9843	0.9830	0.9812	0.9800
0.9930	0.9927	0.9923	0.9919	0.9913	0.9905	0.9896	0.9884	0.9868	0.9856
0.9960	0.9958	0.9955	0.9951	0.9946	0.9940	0.9933	0.9923	0.9908	0.9897
0.9978	0.9976	0.9974	0.9971	0.9968	0.9963	0.9958	0.9949	0.9937	0.9927
0.9989	0.9987	0.9986	0.9984	0.9981	0.9978	0.9974	0.9968	0.9958	0.9949
0.9995	0.9993	0.9992	0.9991	0.9990	0.9987	0.9984	0.9980	0.9972	0.9965
0.9998	0.9997	0.9996	0.9995	0.9994	0.9993	0.9991	0.9988	0.9982	0.9976
0.9999	0.9998	0.9998	0.9998	0.9997	0.9996	0.9995	0.9993	0.9989	0.9984
1.0	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996	0.9993	0.9990
1.0	1.0	1.0	0.9999	0.9999	0.9999	0.9998	0.9998	0.9996	0.9994
1.0	1.0	1.0	1.0	1.0	0.9999	0.9999	0.9999	0.9998	0.9996
1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9999	0.9999	0.9998
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9999	0.9999
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 2

f_1	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.0854
$\zeta^{(1)}$	0.2204	0.2375	0.2542	0.2706	0.2868	0.3028	0.3188	0.3348	0.3510	0.3601
$H^{(1)}$	2.5919	2.5384	2.4903	2.4449	2.4014	2.3599	2.3196	2.2802	2.2403	2.2173
$F^{(1)}$	0.4408	0.3847	0.3293	0.2750	0.2219	0.1701	0.1197	0.0708	0.0239	0
$\varepsilon^{(1)}$	0	0.0010	0.0028	0.0056	0.0097	0.0150	0.0217	0.0300	0.0402	0.0472
f_1	−0.01	−0.02	−0.03	−0.04	−0.05	−0.06	−0.07	−0.08	−0.0835	−0.0852
$\zeta^{(1)}$	0.2034	0.1851	0.1662	0.1462	0.1249	0.1015	0.0746	0.0397	0.0211	0
$H^{(1)}$	2.6441	2.7063	2.7754	2.8538	2.9458	3.0575	3.2051	3.4410	3.5974	3.8150
$F^{(1)}$	0.4997	0.5585	0.6189	0.6807	0.7444	0.8099	0.8779	0.9500	0.9770	0.9909
$\varepsilon^{(1)}$	0.0018	0.0034	0.0067	0.0113	0.0179	0.0263	0.0371	0.0521	0.0591	0.0633

If the point $x = 0$ corresponds to the frontal critical point of the body, at which $U = 0$, then this point is singular, and $F^{(1)}$ will vanish there. From Table 2 it follows that at this point $(x = 0)$

$$f_1 = f_{10} = 0.0854, \qquad z_0^{**} = 0.0854 / U_0' \tag{3.4}$$

which gives the initial value z^{**} in the numerical intergration of Equation (3.3). If, however, as occurs at the leading edge of a plate or a profile with zero nose angle, $U \neq 0$, then $z^{**} = z_0^{**} = 0$, and $\delta_0^{**} = 0$. Finally, in the general case, if $U \neq 0$ for a certain value $x = x_0 > 0$, then the initial condition is $z^{**} = z_0^{**} > 0$ when $x = x_0 > 0$, where $z_0^{**} = \delta_0^{**2}/\nu$ expresses in approximate, summary form the previous history of the development of the boundary layer in the segment $0 \leqslant x \leqslant x_0$. This value z_0^{**} can be used for determining the constant of integration in the first order equation (3.3).

We can set up the following simple and practically convenient computational method for integrating (3.3). As is evident from Fig.3, the curve $F^{(1)}(f_1)$ deviates only slightly from its tangent passing through the point $f_1 = 0$, and therefore can be represented by Equation

$$F^{(1)}(f_1) = a - bf_1 + \varepsilon(f_1) \tag{3.5}$$

where $\varepsilon(f_1)$ expresses the deviation of the curve from its tangent; the magnitude of this deviation is shown in Table 2 and in Fig.3. The numerical values of the constants a and b, as is demonstrated in the following Section, can be

$$a = 0.4408, \qquad b = 5.714 \tag{3.6}$$

Carrying out the formal integration of Equation (3.3), we can obtain one of the following two integral relations:

$$\begin{aligned} f_1(x) &= \frac{U'(x)}{U^b(x)} \int_0^x U^{b-1}(x)\ \{a + \varepsilon\,[f_1(x)]\}\, dx \\ z^{**}(x) &= \frac{1}{U^b(x)} \int_0^x U^{b-1}(x)\ \{a + \varepsilon\,[f_1(x)]\}\, dx \end{aligned} \tag{3.7}$$

in which the constant of integration is chosen from the condition for finiteness of f_1 and z^{**} when $U = 0$.

As is evident from Table 2, the values of ε are small in comparison with the quantity a. Calculation of $f_1(x)$ and $z^{**}(x)$ could be obtained by successive quadratures, starting by neglecting ε in comparison with a

$$f_1(x) = \frac{aU'(x)}{U^b(x)} \int_0^x U^{b-1}\, dx \tag{3.8}$$

but it is simpler to proceed differently. Let us introduce the notation $\varepsilon_k = \varepsilon\,[f_1(x_k)]$, where x_k are the abscissas of points of arbitrary division of the interval x. Then, replacing the actual distribution $\varepsilon(x)$ by a step-function, let us rewrite the second of Equations (3.7) in the form of a recurrence relation

$$U^b(x_k)\, z^{**}(x_k) = U^b(x_{k-1})\, z^{**}(x_{k-1}) + (a + \varepsilon_{k-1}) \int_{x_{k-1}}^{x_k} U^{b-1}(x)\, dx \tag{3.9}$$

enabling us, with the help of previously prepared tables of powers of numbers with positive exponents b and $b - 1$, easily to find z^{**}. Close to those values x_k which correspond to very small f_1, we can moreover simply use the quadrature (3.8). An example of the calculation is given in Section 5.

The numerical integration of the universal equations with two or more form-parameters involves considerable difficulties, since it needs the use of powerful electronic computers.

Comparisons with exact solutions show that the one-parameter approximation represents the main part of the solution. Assuming that at least outside the region immediately adjacent to the point of separation we can take the correction introduced by the two- and three-parameter approximations to be small in comparison with the main part of the solution, let us content ourselves with expressions of these corrections, calculated with the help of series expanded in powers of the form-parameters. The coefficients in these series are expressed as functions of the reduced coordinate ξ , being integrals of a system of ordinary linear differential equations, more easily amendable to numerical integration on computers than the two- or three-parameter universal partial differential equations.

4. Construction of the solution of the multi-parameter universal equation in the form of a series of powers of the form-parameters. We shall seek the solution of the general universal equation (2.8) in the form of a power series [5]

$$\Phi(x, \xi) = \Phi_0(\xi) + \Phi_1(\xi) f_1 + \Phi_{11}(\xi) f_1^2 + \Phi_2(\xi) f_2 + \\ + \Phi_{111}(\xi) f_1^3 + \Phi_{12}(\xi) f_1 f_2 + \Phi_3(\xi) f_3 + \ldots \quad (4.1)$$

First of all let us expand the quantities ζ, H and F in power series with respect to the form-parameters

$$\zeta = \zeta_0 + \zeta_1 f_1 + \zeta_{11} f_1^2 + \zeta_2 f_2 + \zeta_{111} f_1^3 + \zeta_1 f_2 f_1 f_2 + \zeta_3 f_3 + \ldots$$

$$H = H_0 + H_1 f_1 + H_{11} f_1^2 + H_2 f_2 + H_{111} f_1^3 + H_{12} f_1 f_2 + H_3 f_3 + \ldots \quad (4.2)$$

$$F = F_0 + F_1 f_1 + F_{11} f_1^2 + F_2 f_2 + F_{111} f_1^3 + F_{12} f_1 f_2 + F_3 f_3 + \ldots$$

and let us note that the constant coefficients $\zeta_{ij\ldots}$ and $H_{ij\ldots}$, in accordance with Formulas (1.6) and (4.1),(1.12), (bearing in mind that a dot denotes differentiation with respect to ξ) are given by

$$\zeta_{ij\ldots} = B_0 \ddot{\Phi}_{ij\ldots}(0), \qquad H_{ij\ldots} = -\frac{1}{B_0}\Phi_{ij\ldots}(\infty) \quad (4.3)$$

after which the constants $F_{ij\ldots}$ according to the first of Formulas (1.4) are determined thus:

$$F_0 = 2\zeta_0, \quad F_1 = 2(\zeta_1 - H_0 - 2), \quad F_{11} = 2(\zeta_{11} - H_1), \quad F_2 = 2\zeta_2 \\ F_{111} = 2(\zeta_{111} - H_{11}), \quad F_{12} = 2(\zeta_{12} - H_2), \quad F_3 = 2\zeta_3, \ldots \quad (4.4)$$

Substituting these expansions in Equation (2.8) and equating coefficients in similar one-term powers of the complex form-parameters, we obtain the following system of ordinary linear nonhomogeneous differential equations [5] for the unknown functions $\Phi_{ij\ldots}(\xi)$:

$$L_k(\Phi_{ij\ldots}) = -(1/B_0^2)\,\zeta_{ij\ldots}\Phi_0\ddot{\Phi}_0 + \Gamma_{ij\ldots}$$

$$\Phi_{ij\ldots} = \dot{\Phi}_{ij\ldots} = 0 \quad \text{for } \xi = 0, \qquad \dot{\Phi}_{ij\ldots} \to 0 \quad \text{for } \xi \to \infty \quad (4.5)$$

$$(k = i + j + \ldots;\ i, j, \ldots = 1, 2, \ldots)$$

Here by L_k we understand the linear operator

$$L_k = D^3 + \Phi_0 D^2 - 2k\dot{\Phi}_0 D + (2k+1)\ddot{\Phi}_0, \qquad D = d/d\xi \quad (4.6)$$

The Function $\Phi_0(\xi)$ satisfies Blasius' equation (1.8), whilst the functions $\Gamma_{ij\ldots}(\xi)$, appearing in the right-hand side of the equation with the same indices k as the operator L_k, are expressed in terms of the functions $\Phi_{ij\ldots}(\xi)$ and the constants $\zeta_{ij\ldots}$, $H_{ij\ldots}$, already calculated in the integration of equations with the index of the operator less than k .

We reproduce the expressions for $\Gamma_{ij\ldots}(\xi)$, corresponding to indices

k from 1 to 3

$$\Gamma_1 = (1 / B_0^2) [\Phi_0'^2 - 1 + (H_0 + 1) \Phi_0\Phi_0''] \tag{4.7}$$

$$\Gamma_{11} = 2\Phi_1'^2 - 3\Phi_1\Phi_1'' + (1 / B_0^2) [H_1\Phi_0\Phi_0'' - (\zeta_1 - H_0 - 1) \Phi_0\Phi_1'' - (3\zeta_1 - 3H_0 - 5) \Phi_0''\Phi_1 + 2 (\zeta_1 - H_0 - 1) \Phi_0'\Phi_1']$$

$$\Gamma_2 = (1 / B_0^2) (\Phi_0'\Phi_1' - \Phi_0''\Phi_1)$$

$$\Gamma_{111} = 6\Phi_1'\Phi_{11}' - 3\Phi_1\Phi_{11}'' - 5\Phi_1''\Phi_{11} + (1 / B_0^2) [(2\zeta_1 - 2H_0 - 3)\times(\Phi_1'^2 + 2\Phi_0\Phi_{11}) + H_{11}\Phi_0\Phi_0'' - (\zeta_{11} - H_1) (\Phi_0\Phi_1'' - 2\Phi_0'\Phi_1' + 3\Phi_0''\Phi_1) - (\zeta_1 - H_0 - 1) \Phi_0\Phi_{11}'' - (3\zeta_1 - 3H_0 - 5) \Phi_1\Phi_1'' - (5\zeta_1 - 5H_0 - 9) \Phi_0''\Phi_{11}]$$

$$\Gamma_{12} = - 3\Phi_1\Phi_2'' + 6\Phi_1'\Phi_2' - 5\Phi_1''\Phi_2 - (1 / B_0^2) [(\zeta_1 - H_0 - 1) \Phi_0\Phi_2'' + (5\zeta_1 - 5H_0 - 8) \Phi_0''\Phi_2 - H_2\Phi_0\Phi_0'' - (4\zeta_1 - 4H_0 - 5) \Phi_0'\Phi_2' - \Phi_1'^2 + \Phi_1\Phi_1'' - 2\Phi_0'\Phi_{11}' + 2\Phi_0''\Phi_{11} - \zeta_2 (2\Phi_0'\Phi_1' - 3\Phi_0''\Phi_1 - \Phi_0\Phi_1'')]$$

$$\Gamma_3 = (1 / B_0^2) (\Phi_0'\Phi_2' - \Phi_0''\Phi_2)$$

The peculiarity of Equations (1.5) and (2.8) noticed in Sections 1 and 2 survives also in the system of equations (4.5), since, according to the first of Equations (4.3), the quantities $\zeta_{ij\ldots}$, appearing in the right-hand side of the equations of system (4.5), require for their determination a prior knowledge of the functions $\Phi_{ij\ldots}(\xi)$, being the solutions of the same equations. Thanks to the linearity of the system (4.5), this difficulty is easily eliminated. Setting in system (4.5)

$$\Phi_{ij\ldots} (\xi) = X_{ij\ldots} (\xi) + \zeta_{ij\ldots}Y_{ij\ldots} (\xi) \tag{4.8}$$

we arrive at a combination of two linear nonhomogeneous systems

$$L_k (X_{ij\ldots}) = \Gamma_{ij\ldots}, \qquad L_k (Y_{ij\ldots}) = - (1 / B_0^2) \Phi_0\Phi_0'' \tag{4.9}$$

with the same zero boundary conditions for the functions $X_{ij\ldots}(\xi)$ and $Y_{ij\ldots}(\xi)$ as for the functions $\Phi_{ij\ldots}(\xi)$ in system (4.5).

After the functions $X_{ij\ldots}(\xi)$ and $Y_{ij\ldots}(\xi)$ are determined and once and for all tabulated, the quantities $\zeta_{ij\ldots}$ can be calculated according to Formulas

$$\zeta_{ij\ldots} = \frac{B_0X_{ij\ldots}''(0)}{1 - B_0Y_{ij\ldots}''(0)} \tag{4.10}$$

The numerical integration of the system of equations (4.9) was effected in the Leningrad Computation Center of the Academy of Sciences, USSR on the БЭСМ-2 (BESM-2) computer. Tables were constructed of the functions X_1, X_{11}, X_2, X_{111}, X_{12}, X_3 and Y_1, $Y_{11} = Y_2$, $Y_{111} = Y_{12} = Y_3$, and from them were determined also values of the constants ζ_1, ζ_{11}, ζ_2, ζ_{111}, ζ_{12}, ζ_3; H_1, H_{11}, H_2, H_{111}, H_{12}, H_3; F_1, F_{11}, F_2, F_{111}, F_{12}, F_3.

Since it is not possible to publish the tables of the functions $X_{ij\ldots}(\xi)$ and $Y_{ij\ldots}(\xi)$, we reproduce those expansions (4.2), which are the most important for practical application, using the numerical values of the coefficients $\zeta_{ij\ldots}$, $H_{ij\ldots}$, $F_{ij\ldots}$

The following values were obtained:

$$\zeta = 0.2204 + 1.7350f_1 - 2.4188f_1^2 - 0.2992f_2 + 18.234f_1^3 - 0.1653f_1f_2 + 0.0937f_3 + \ldots$$
$$H = 2.5919 - 5.4282f_1 + 21.914f_1^2 + 1.4741f_2 - - 163.06f_1^3 - 4.8076f_1f_2 - 0.50613f_3 + \ldots \quad (4.11)$$
$$F = 0.4408 - 5.7139f_1 + 6.0189f_1^2 - 0.5984f_2 - 7.3611f_1^3 - 3.2753f_1f_2 - 1.0123f_3 + \ldots$$

Increasing slightly the speed of convergence, let us isolate in these series the already tabulated one-parameter portions which correspond to condition $f_2 = f_3 = \ldots = 0$. Then we obtain the following formulas for allowing for the effect of the second and third form-parameters:

$$\zeta = \zeta^{(1)}(f_1) - 0.2992f_2 - 0.1653f_1f_2 + 0.0937f_3 + \ldots$$
$$H = H^{(1)}(f_1) + 1.4741f_2 - 4.8076f_1f_2 - 0.50613f_3 + \ldots \quad (4.12)$$
$$F = F^{(1)}(f_1) - 0.5984f_2 - 3.2753f_1f_2 - 1.0123f_3 + \ldots$$

which, after the two-parameter universal equation has been integrated, can be substituted for greater accuracy.

As is shown by the first comparative calculations, for determining the quantity $z^{**}(x)$, and consequently, the momentum thickness $\delta^{**}(x)$, it is sufficient to use the simple method, described at the end of Section 3, only in certain cases, possibly allowing for the effect of subsequent form-parameters in the correction ϵ .

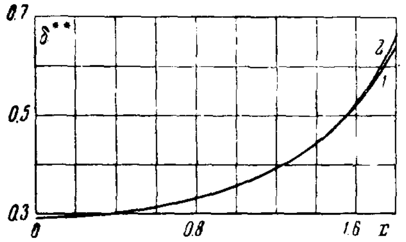

Fig. 4

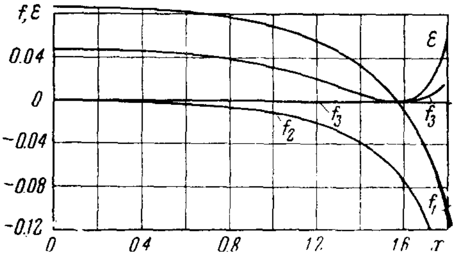

Fig. 5

5. An example of the calculation and comparison with the exact solution. To illustrate the arguments propounded in the preceding Sections, let us consider the case of a sinusoidal velocity distribution at the outer boundary calculated by Terrill [10], which corresponds to the streamline flow past a circular cylinder of an irrotational stream of ideal incompressible fluid.

Let us agree in the present Section to deal with the dimensionless quantities, obtained by dividing the dimensional longitudinal lengths and velocities by the radius of the cylinder and the velocity at infinity, respectively, and the transverse ones by the same quantities, but decreased by the factor $\sqrt{R}$, where the Reynolds number R is constructed from the radius of the cylinder and the flow velocity.

In these dimensionless quantities we shall have

$$U = \sin x, \qquad z^{**} = \delta^{**2}, \qquad \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\zeta U}{\delta^{**}} \quad (5.1)$$

In Fig.4 we present for comparison two curves of the dimensionless quantity $\delta^{**}(x)$: one calculated by the method described at the end of Section 3 – without correction for the effect of the second and third form-parameters, whilst the other corresponds to the exact solution. In general nature of the growth of the quantity δ^{**}, especially close to the separation, there is scarcely any increase in the relative accuracy of the calculation (in Fig. 4 the maximal discrepancy at the point of separation does not exceed 3 per cent).

In Fig.5 are given curves of the variation along the boundary layer of the values of the three form-parameters

$$f_1(x) = z^{**} \cos x, \qquad f_2(x) = -z^{**2} \sin^2 x, \qquad f_3(x) = -z^{**3} \sin^2 x \cos \qquad (5.2)$$

We can observe the completely insignificant value of the form-parameter f_3 everywhere except in the neighborhood of the point of separation, and the rather small value of f_2 in the accelerating part of the layer. Probably this explains the success of the application of the one-parameter method to the majority of practical calculation .

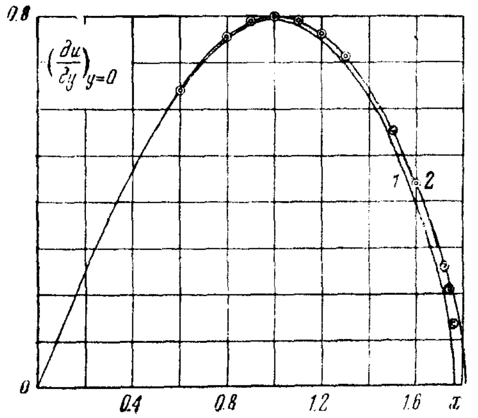

Fig. 6

Fig.6 shows the distribution of the dimensionless quantity

$$(\partial u / \partial y)_{y=0},$$

characterizing the local coefficient of friction at the surface of the cylinder. As is obvious from the above, the one-parameter approximation (1) is sufficient to obtain a satisfactory result. Inclusion of only the second term (succeeding terms are very small) even in the first of Equations (4.12) produces almost complete agreement (in Fig. 6 indicated by the circles) with the exact solution (2) everywhere except at the region of separation, where, evidently, the corrections based on the use of the power series are no longer applicable. The method explained in the proceding Sections is applicable not only to the isothermal boundary layer in an incompressible fluid, but also allows immediate generalization to the case of the boundary layer in a gaseous stream with large sub- and supersonic velocities, when it has become impossible to neglect the effect of compressibility of the gas, processes involving the liberation and propagation of heat, and also the occurance of dissociation of the molecules of the gas.

6. The universal equations of the boundary layer in a stream of homogeneous gas with large velocities. Let us agree in what follows to denote by the indices: a, the thermodynamic quantities for adiabatically and isentropically retarded gas inside the boundary layer; 1, the same, but outside and on the outer surface of the boundary layer; e and w, the dynamic and thermodynamic quantities on the outer surface of the layer and on the wetted surface of the body. Let us for simplicity take a linear law connecting the dynamic coefficient of viscosity of the gas μ with the enthalpy $h = c_p T$

$$\mu / \mu_1 = Ch / h_1, \qquad C = (T_w / T_1)^{1/2} (T_1 + T_s) / (T_w + T_s) \qquad (6.1)$$

Here T_s is the constant in Sutherland's formula.

Let us pass, in the boundary layer equations, from the usual physical coordinates and velocities x, y, u, v to new variables X, Y, U, V, by means of the Dorodnitsyn-Stewartson transformation (ρ is the density of the gas, M_e is the Mach number at the outer surface)

$$X = \int_0^x C \chi_e^{\frac{3k-1}{2(k-1)}} dx, \qquad Y = \chi_e^{\frac{k+1}{2(k-1)}} \int_0^y \frac{\rho}{\rho_e} dy, \qquad k = \frac{c_p}{c_v} \qquad (6.2)$$

$$U = \chi_e^{-1/2}u, \quad V = \chi_e^{-1/2}v^\circ, \quad \chi_e = \frac{h_e}{h_1} = 1 - \varkappa, \quad \varkappa = \frac{u_e^2}{2h_1} \qquad \text{(6.2) (cont.)}$$

$$v^\circ = \frac{1}{C}\chi_e^{-\frac{3k-1}{2(k-1)}}\left[u\frac{\partial Y}{\partial x} + \left(\frac{\rho v}{\rho_e}\right)\chi_e^{\frac{k+1}{2(k-1)}}\right]$$

We shall introduce, moreover, the stream function $\Psi(X, Y)$

$$U = \partial\Psi / \partial Y, \qquad V = -\partial\Psi / \partial X \tag{6.3}$$

and the heat function

$$S = (h_a / h_1) - 1 \tag{6.4}$$

Then we obtain [11] the well-known system of equations (ν is the kinematic coefficient of viscosity of the gas, σ is the Prandtl number)

$$\frac{\partial\Psi}{\partial Y}\frac{\partial^2\Psi}{\partial X\,\partial Y} - \frac{\partial\Psi}{\partial X}\frac{\partial^2\Psi}{\partial Y^2} = U_e\frac{dU_e}{dX}(1+S) + \nu_1\frac{\partial^3\Psi}{\partial Y^3}$$

$$\frac{\partial\Psi}{\partial Y}\frac{\partial S}{\partial X} - \frac{\partial\Psi}{\partial X}\frac{\partial S}{\partial Y} = \frac{\nu_1}{\sigma}\left\{\frac{\partial^2 S}{\partial Y^2} - (1-\sigma)\varkappa\frac{\partial^2}{\partial Y^2}\left[\left(\frac{1}{U_e}\frac{\partial\Psi}{\partial Y}\right)^2\right]\right\}$$

$$\Psi = \frac{\partial\Psi}{\partial Y} = 0, \quad S = S_w \quad \text{for } Y = 0, \qquad \frac{\partial\Psi}{\partial Y} \to U_e, \quad S \to 0 \quad \text{for } Y \to \infty \tag{6.5}$$

$$\frac{\partial\Psi}{\partial Y} = U_0(Y), \quad S = S_0(Y) \quad \text{for } X = X_0$$

We shall introduce for consideration the following two conventional thicknesses of the boundary layer in the variables of (6.2): the displacement thickness Δ^* and the momentum thickness Δ^{**}

$$\Delta^* = \int_0^\infty \left[1 - \frac{U}{U_e} + S\right] dY, \qquad \Delta^{**} = \int_0^\infty \frac{U}{U_e}\left[1 - \frac{U}{U_e}\right] dY \tag{6.6}$$

Then the equation of momentum, easily derived from (6.5), preserves the same form as in incompressible fluid (a prime denotes differentiation with respect to X)

$$\frac{dZ^{**}}{dX} = \frac{F}{U_e}, \qquad \frac{df}{dX} = \frac{U_e'}{U_e}F + \frac{U_e''}{U_e'}f \tag{6.7}$$

Here

$$Z^{**} = \frac{\Delta^{**2}}{\nu_1}, \quad f = \frac{U_e'\Delta^{**2}}{\nu_1}, \quad H = \frac{\Delta^*}{\Delta^{**}}, \quad \zeta = \left[\frac{\partial(U/U_e)}{\partial(Y/\Delta^{**})}\right]_{Y=0} \tag{6.8}$$

$$F = 2\,[\zeta - (2+H)f]$$

If now, completely repeating the processes described in the foregoing Sections, we pass from the variables X, Y, Ψ to the new "reduced" variables

$$X = X, \qquad \xi = B_0 Y / \Delta^{**}, \qquad \Phi = B_0\Psi / (U_e\Delta^{**}) \tag{6.9}$$

introduce an infinite combination of form-parameters

$$f_0 = \varkappa, \quad f_k = U_e^{k-1}(d^k U_e / dX^k)\, Z^{**k}; \quad f_1 = f \quad (k = 1, 2, \ldots) \tag{6.10}$$

and take them together with the reduced ordinate ξ as the new combination of independent variables, then the system of equations (6.5) takes the following form, derived by S.M. Kapustianskii, post-graduate of the Leningrad

Polytechnic Institute: (6.11)

$$\frac{\partial^3\Phi}{\partial\xi^3} + \frac{F+2f_1}{2B_0^2}\Phi\frac{\partial^2\Phi}{\partial\xi^2} + \frac{f_1}{B_0^2}\left[1-\left(\frac{\partial\Phi}{\partial\xi}\right)^2 + S\right] = \frac{1}{B_0^2}\sum_{k=0}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi\,\partial f_k} - \frac{\partial\Phi}{\partial f_k}\frac{\partial^2\Phi}{\partial\xi^2}\right)$$

$$\frac{\partial^2 S}{\partial\xi^2} + \sigma\frac{F_1+2f_1}{2B_0^2}\Phi\frac{\partial S}{\partial\xi} + 2(\sigma-1)\varkappa\frac{\partial}{\partial\xi}\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi^2}\right) = \frac{\sigma}{B_0^2}\sum_{k=0}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial S}{\partial f_k} - \frac{\partial\Phi}{\partial f_k}\frac{\partial S}{\partial\xi}\right)$$

$$\Phi = \frac{\partial\Phi}{\partial\xi} = 0,\quad S = S_w \quad \text{for } \xi = 0; \qquad \frac{\partial\Phi}{\partial\xi}\to 1,\quad S\to 0 \quad \text{for } \xi\to\infty$$

$$\Phi = \Phi_0(\xi),\quad S = S_0(\xi) \quad \text{for } f_0 = \varkappa_0 = u^2_\infty/2h_1,\quad f_1 = f_2 = \ldots = 0$$

In this system the quantities θ_k are defined just as in Equation (2.6), but are expressed in terms of the new values of F and f_k, given by Equations (6.8) and (6.10), whilst $\theta_0 = 2\varkappa(\varkappa-1)f_1$. Regarding the normalizing factor B_0, this, as previously, is chosen from the conditions that when $f_1 = f_2 = \ldots = 0$, Equation (6.11) transforms into the self-similar solution of Blasius' problem $\Phi_0(\xi)$ for the plate in the presence of heat transfer and dissipation (the dot is the sign of differentiation with respect to ξ)

$$\Phi_0^{\cdots} + \Phi_0\Phi_0^{\cdot\cdot} = 0 \tag{6.12}$$

$$S_0^{\cdot\cdot} + \sigma\Phi_0 S_0^{\cdot} + 2(\sigma-1)\varkappa_0(\Phi_0^{\cdot\cdot 2} + \Phi_0^{\cdot}\Phi_0^{\cdots}) = 0,\quad \varkappa_0 = u_\infty^2/(2h_1)$$

$$\Phi_0 = \Phi_0^{\cdot} = 0,\quad S_0 = S_w, \quad \text{for } \xi = 0,\quad \Phi_0^{\cdot}\to 1,\quad S_0\to 0 \quad \text{for } \xi\to\infty$$

Thanks to the assumed linearity of the relation between viscosity and temperature, $\Phi_0(\xi)$ does not differ at all from the corresponding function in incompressible liquid, and consequently B_0 also is the same. It should not be forgotten, however, that H_0, according to the accepted definition (6.6), becomes now different. We shall have

$$B_0 = \int_0^\infty \frac{\partial\Phi}{\partial\xi}\left(1-\frac{\partial\Phi}{\partial\xi}\right)d\xi = \int_0^\infty \Phi_0^{\cdot}(1-\Phi_0^{\cdot})\,d\xi$$

$$H_0 = \frac{\Delta^*}{\Delta^{**}} = \frac{1}{B_0}\int_0^\infty(1-\Phi_0^{\cdot}+S_0)\,d\xi,\qquad \zeta_0 = B_0\Phi_0^{\cdot\cdot}(0) \tag{6.13}$$

Equations (6.11) contain a number of parameters characterizing the conditions of actual individual problems. First of all, this is the assumed constant value of Prandtl's number σ. Such also is the dimensionless quantity $S_w = (h_w/h_1) - 1$, expressed in terms of the actual value of the "temperature factor" T_w/T_∞ and the Mach number M_∞ of the free stream, which are characteristic of each individual problem, according to Formula

$$S_w = (T_w/T_\infty)\,[1+{}^1/_2(k-1)M_\infty^2]^{-1} - 1 \tag{6.14}$$

The local "compressibility factor" of the gas $\varkappa$, depending on the "local" Mach numbers $M_e = u_e/a_e$ or $M_e^* = u_e/a^*$, (where a^* is the critical velocity of the gas outside the boundary layer) and in the case of homogeneous gas on the physical constant of gas $k = c_p/c_v$, according to Formulas

$$\varkappa = \frac{k-1}{2} M_e^2 \Big/ \left(1 + \frac{k-1}{2} M_e^2\right) = \frac{k-1}{k+1} M_e^{*2}$$

appears in a number of arguments of (6.10) and does not violate the universality of the system (6.11). Moreover we retain the possibility of carrying out in advance once and for all the integration of the system (6.11) for different values of the parameters σ, S_w and also $\varkappa$ $(0 \leqslant \varkappa \leqslant 1)$. The latter is permissible if in Equations (6.11) in the first approximation the differentiation with respect to $\varkappa$ is dropped. The term "universal" for the system of equations (6.11) can be retained in a certain more restricted sense than before.

Retaining the terms "one-", "two-parametric", and so on, depending on the number of form-parameters f_k, we obtain with $\sigma = 1$ a "one-parametric" system

$$\frac{\partial^3\Phi^{(1)}}{\partial\xi^3} + \frac{F^{(1)}+2f_1}{2B_0^2}\Phi^{(1)}\frac{\partial^2\Phi^{(1)}}{\partial\xi^2} + \frac{f_1}{B_0^2}\left[1-\left(\frac{\partial\Phi^{(1)}}{\partial\xi}\right)^2 + S^{(1)}\right] = \tag{6.15}$$

$$= \frac{1}{B_0^2}F^{(1)}f_1\left(\frac{\partial\Phi^{(1)}}{\partial\xi}\frac{\partial^2\Phi^{(1)}}{\partial\xi\partial f_1} - \frac{\partial\Phi^{(1)}}{\partial f_1}\frac{\partial^2\Phi^{(1)}}{\partial\xi^2}\right)$$

$$\frac{\partial^2 S^{(1)}}{\partial\xi^2} + \frac{F^{(1)}+2f_1}{2B_0^2}\Phi^{(1)}\frac{\partial S^{(1)}}{\partial\xi} = \frac{1}{B_0^2}F^{(1)}f_1\left(\frac{\partial\Phi^{(1)}}{\partial\xi}\frac{\partial S^{(1)}}{\partial f_1} - \frac{\partial\Phi^{(1)}}{\partial f_1}\frac{\partial S^{(1)}}{\partial\xi}\right)$$

$$\Phi^{(1)} = \Phi^{\cdot(1)} = 0,\quad S^{(1)} = S_w \quad \text{for } \xi = 0,\qquad \Phi^{\cdot(1)} \to 1,\quad S^{(1)} \to 0 \quad \text{for } \xi \to \infty$$

$$\Phi^{(1)} = \Phi_0(\xi),\quad S^{(1)} = S_0(\xi) \quad \text{for } f_1 = 0$$

The tables already compiled for the one-parametric approximation of the universal equation (6.11) by means of the numerical integration of Equations (6.15), should agree with the earlier published one-parametric method of Cohen and Reshetko [12]. The latter method is based on the intuitive assumption of the suitability of using for approximating functions a class of exact solutions, corresponding to the power of the stipulated external velocity [11], and derives the relation of this to the decrease of friction and the onset of separation. Moreover, the fundamental calculational conveniences of the method occur only in a narrow range of values of the parameter S_w.

7. The boundary layer in gas in dissociated equilibrium. In the case of the boundary layer in gas in dissociated equilibrium the universal equations become yet more complex on account of the emergence of a number of new factors. First of all, in this case the connection between the dynamic coefficient of viscosity and absolute temperature can no longer be taken as linear, and we have to adopt the general nonlinear relation of Sutherland (up to 4000 °K) and thereafter use special formulas. Moreover, the quantities c_p and σ are no longer constant, and the density ratio $\rho/\rho_1 = \rho^\circ$ cannot now be expressed, as it was in the nondissociated gas, by means of the inverse ratio of the corresponding temperatures, but requires special values from tables of the termodynamic functions for air or other gas as a function of the dimensionless enthalpy $h = h^\circ/h_1$ and the pressure p. The same applies also to the quantity $N = \mu\rho/(\mu_w\rho_w)$, which in the case of a homogeneous, nondissociated gas and for a linear viscosity law would be equal to unity, whilst in the present case it is a function of the dimensionless enthalpy and pressure.

If we ignore the influence of pressure, which in a wide range of ratios of air pressure p to the atmospherec pressure p_a $(10^{-4} \leqslant p/p_a \leqslant 10)$ is small, then we can assume that $N = N(h^\circ/h^\circ_w)$ and for a given M_∞ that $\rho^\circ = \rho^\circ(h^\circ)$. Moreover, for air in dissociative equilibrium up to temperatures of order 9 000°K we can take the Prandtl number σ as constant, whilst the Lewis number is also constant and equal to unity. With these simplifying but fully plausible assumptions, and using dimensionless variables

$$X = \frac{1}{\mu^* \rho^*} \int_0^x \mu_w \rho_w dx, \qquad Y = \int_0^y \frac{\rho}{\rho^*} dy \tag{7.1}$$

where the asterisk superscript denotes quantities referred to an arbitrary given state of gas, with the existence of a stream function Ψ defined by the system of equations

$$u = \frac{\partial \Psi}{\partial Y}, \qquad v^\circ = \frac{\mu^* \rho^*}{\mu_w \rho} \left[u \left(\frac{\partial Y}{\partial x} \right) + v \left(\frac{\rho}{\rho^*} \right) \right] = -\frac{\partial \Psi}{\partial X} \tag{7.2}$$

The boundary layer equations can be represented in the form

$$\frac{\partial \Psi}{\partial Y} \frac{\partial^2 \Psi}{\partial X \partial Y} - \frac{\partial \Psi}{\partial X} \frac{\partial^2 \Psi}{\partial Y^2} = \frac{\rho_e}{\rho} u_e \frac{du_e}{dX} + \nu^* \frac{\partial}{\partial Y} \left(N \frac{\partial^2 \Psi}{\partial Y^2} \right)$$
$$\frac{\partial \Psi}{\partial Y} \frac{\partial h}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial h}{\partial Y} = -\frac{\rho_e}{\rho} u_e \frac{du_e}{dX} \frac{\partial \Psi}{\partial Y} + \nu^* N \left(\frac{d^2 \Psi}{dY^2} \right)^2 + \nu^* \frac{\partial}{\partial Y} \left(\frac{N}{\sigma} \frac{\partial h}{\partial Y} \right) \tag{7.3}$$

$$\Psi = \frac{\partial \Psi}{\partial Y} = 0, \quad h = h_w \quad \text{for} \quad Y = 0; \qquad \frac{\partial \Psi}{\partial Y} \to u_e(X), \quad h \to h_e(X) \quad \text{for} \quad Y \to \infty$$
$$\frac{\partial \Psi}{\partial Y} = u_0(Y), \quad h = h_0(Y) \quad \text{for} \quad X = X_0$$

The density ratio ρ_e/ρ appearing in the right-hand sides of both equations of the system (7.3), is expressed in the terms of the given functions ρ° and $\varkappa$, thus:

$$\rho_e / \rho = (\rho_e / \rho_1)(\rho_1 / \rho) = \rho^\circ (h_e^\circ) / \rho^\circ (h^\circ) \tag{7.4}$$

In accordance with the notation (6.2) the numerator of this fraction is expressed in terms of $\varkappa$

$$\rho^\circ (h_e^\circ) = \rho^\circ (\chi_e) = \rho^\circ (1 - \varkappa) \tag{7.5}$$

The conversion from the system of equations (7.3) to the universal system was carried out by N.V. Krivtsov (post-graduate of the Leningrad Polytechnic Institute) just as in Section 6, or in the preceding Sections for the case of the isothermal boundary layer in incompressible fluid.

He introduces the same set of form-parameters with respect to the external form

$$f_0 = \varkappa, \; f_k = u_e^{k-1} (d^k u_e / dX^k) Z^{**k}, \qquad Z^{**} = \Delta^{**2} / \nu_* \quad (k = 1, 2, \ldots) \tag{7.6}$$

and shows that if we take for the definitions of the displacement thickness Δ^* and the momentum thickness Δ^{**} Expressions

$$\Delta^* = \int_0^\infty \left(\frac{\rho_e}{\rho} - \frac{u}{u_e} \right) dY, \qquad \Delta^{**} = \int_0^\infty \frac{u}{u_e} \left(1 - \frac{u}{u_e} \right) dY \tag{7.7}$$

then the momentum equation will have the same form as in the case of the incompressible fluid (primes denote differentiation with respect to X)

$$\frac{dZ^{**}}{dX} = \frac{F}{u_e}, \qquad \frac{df_1}{dX} = \frac{u_e'}{u_e} F + \frac{u_e''}{u_e'} f_1$$
$$F = 2 [\zeta - (2 + H) f_1], \quad \zeta = \left[\frac{\partial (u / u_e)}{\partial (Y / \Delta^{**})} \right]_{Y=0}, \quad H = \frac{\Delta^*}{\Delta^{**}} \tag{7.8}$$

Of the same form also is the recurrence differential equation for the differential of the form-parameter f_k with respect to X, and consequently also the form of the functions $\theta_k (f_1, f_2, \ldots)$.

Transforming in the system (7.3) to the new variables (B_0 is a normalizing factor)

$$X = X, \qquad \xi = B_0 Y / \Delta^{**}, \qquad \Phi = B_0 \Psi / (u_e \Delta^{**}) \tag{7.9}$$

and introducing, just as before, the form-parameters (7.6) into the number of independent variables, we obtain the universal system of equations (in a more restricted sense than in the case of the incompressible fluid, as was pointed out in Section 6) ($\theta_0 = 2\varkappa f_1$)

$$\frac{\partial}{\partial\xi}\left[N(h^\circ)\frac{\partial^2\Phi}{\partial\xi^2}\right]+\frac{F+2f_1}{2B_0^2}\Phi\frac{\partial^2\Phi}{\partial\xi^2}+\frac{f_1}{B_0^2}\left[\frac{\rho^\circ(1-\varkappa)}{\rho^\circ(h^\circ)}-\left(\frac{\partial\Phi}{\partial\xi}\right)^2\right]= \tag{7.10}$$

$$=\frac{1}{B_0^2}\sum_{k=0}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial^2\Phi}{\partial\xi\,\partial f_k}-\frac{\partial\Phi}{\partial f_k}\frac{\partial^2\Phi}{\partial\xi^2}\right)$$

$$\frac{\partial}{\partial\xi}\left[\frac{N(h^\circ)}{\sigma}\frac{\partial h^\circ}{\partial\xi}\right]+\frac{F+2f_1}{2B_0^2}\Phi\frac{\partial h^\circ}{\partial\xi}-2\frac{\varkappa\rho^\circ(1-\varkappa)}{B_0^2}\frac{f_1}{\rho^\circ(h)^\circ}\frac{\partial\Phi}{\partial\xi}+$$

$$+2\varkappa N(h^\circ)\left(\frac{\partial^2\Phi}{\partial\xi^2}\right)^2=\frac{1}{B_0^2}\sum_{k=0}^{\infty}\theta_k\left(\frac{\partial\Phi}{\partial\xi}\frac{\partial h^\circ}{\partial f_k}-\frac{\partial\Phi}{\partial f_k}\frac{\partial h^\circ}{\partial\xi}\right)$$

$$\Phi=\frac{\partial\Phi}{\partial\xi}=0,\quad h^\circ=h_w^\circ\quad\text{for}\quad\xi=0,\qquad\frac{\partial\Phi}{\partial\xi}\to 1,\quad h^\circ\to 1-\varkappa\quad\text{for}\quad\xi\to\infty$$

$$\Phi=\Phi_0(\xi),\quad h^\circ=h_0^\circ(\xi)\quad\text{for}\quad f_0=\varkappa,\quad f_1=f_2=\ldots=0$$

The normalizing constant B_0 is chosen so that the functions $\Phi_0{}^\circ(\xi)$ and $h_0{}^\circ(\xi)$, representing the solution of the self-similar problem, corresponding to constant external velocity ($u_* = u_\infty$), satisfy the system of equations of the form

$$\frac{d}{d\xi}\left[N(h_0^\circ)\frac{d^2\Phi_0}{d\xi^2}\right]+\Phi_0\frac{d^2\Phi_0}{d\xi^2}=0,\qquad\frac{d}{d\xi}\left[\frac{N(h_0^\circ)}{\sigma}\frac{dh_0^\circ}{d\xi}\right]+\Phi_0\frac{dh_0^\circ}{d\xi}+2\varkappa_0N(h_0^\circ)\left(\frac{d^2\Phi_0}{d\xi^2}\right)^2=0 \tag{7.11}$$

$$\Phi_0=\Phi_0'=0,\quad h_0^\circ=h_w^\circ\quad\text{for}\quad\xi=0,\quad\Phi_0'\to 1,\qquad h_0^\circ\to 1-\varkappa\quad\text{for}\quad\xi\to\infty$$

representing the generalization of Blasius' problem to the case of a gas in dissociative equilibrium. Here $\varkappa_0 = u_\infty^2/2h_1$ is a function of only the given Mach number M_∞. The normalizing constant B_0, as in the preceding cases, is determined from Equation $2B_0{}^2 = F_0$ but since, by virtue of the lack of "autonomy" of the first equation of system (7.11), ζ_0 will now depend on h_w, σ and $\varkappa_0$, it is evident that B_0 needs to be calculated for different values of these parameters.

In the universal system of Equations (7.10) the quantities $h_w{}^\circ$, σ and $\varkappa$ play the roles of parameters characterizing the actual individual problems. The parameter $h_w{}^\circ$ is analogous to the parameter S_w of the foregoing Section.

In the more general case of the boundary layer in a gas in nonequilibrium dissociation, there appears in the universal system of equations the diffusion equation also, including as a new unknown the concentration of atoms. Granted a series of plausible simplifying assumptions, even in this case, the universal system can, for a selected combination of values of the physical parameters, be integrated once and for all, while the "reduced" stream function, enthalpy and concentration can be tabulated. The difficulties arising here are of a purely technical, computational character and can be overcome by applying the method of successive approximations.

BIBLIOGRAPHY

1. Falkner, V.M., A further investigation of the boundary layer equations. Aeronaut.Res.Council Repts. and Mem., № 1884, p.1-45, 1937.
2. Görtler, H.A., A new series for the calculation of steady laminar boundary layer flows. J.Math. and Mech., Vol.6, p.1-66, 1957.

3. Shkadov, V.Ia., Ob integrirovanii uravnenii pogranichnogo sloia (Integration of the boundary layer equations). Dokl.Akad.Nauk SSSR, Vol.126, № 4, 1959.

4. Shkadov, V.Ia., K resheniiu zadachi o pogranichnom sloe (Solution of the boundary layer problem). Izv.Akad.Nauk SSSR, OTN, Mekhanika i mashinostroenie, № 3, 1962.

5. Loitsianskii, L.G., Dvizheniia v pogranichnom sloe, blizkie k avtomodel'nym (Nearly self-similar motions in the boundary layer). Trudy L.P.I. Tekhnicheskaia gidromekhanika, № 230, 1964.

6. Loitsianskii, L.G., Laminarn i pogranichnyi sloi (Laminar Boundary Layer) Fizmatgiz, M., 1962.

7. Laminar boundary layers. Fluid Motion Memoirs, ed. by L. Rosenhead, Oxford, 1963.

8. Howarth, L., On the solution of boundary layer equations. Proc.Roy.Soc. London A., Vol.164, p.547, 1938.

9. Mel'nikov, A.P., Laminarnyi pogranichnyi sloi kryla i ego raschet (The laminar boundary layer wing and its design). Trudy leningr.voenno-vozd.Akad., № 1, 1942.

10. Terrill, R.M., Laminar boundary-layer flow near separation with and without suction. Phylos.Trans.Roy.Soc., London A, Vol.253, 1922.

11. Cohen, C.B. and Reshotko, E., Similar solutions for the compressible laminar boundary-layers with heat transfer and pressure gradient. NACA Rept.1293, 1956.

12. Cohen, C.B. and Reshotko, E., The compressible laminar boundary layer with heat transfer and arbitrary pressure gradient. NACA Rept.1294, 1956.

Translated by A.H.A.